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Quantum entanglement of charges in bound states with finite-size dyons

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Abstract: We show that the presence of finite-size monopoles can lead to a number of interesting physical processes involving quantum entanglement of charges. Taking as a model the classical solution of the $N = 2$ SU(2) Yang-Mills theory, we study interaction between dyons and scalar particles in the adjoint and fundamental representation. We find that there are bound states of scalars and dyons, which, remarkably, are always an entangled configuration of the form $|\psi\rangle = |\mathrm{dyon}_+\rangle|\mathrm{scalar}_-\rangle \pm |\mathrm{dyon}_-\rangle|\mathrm{scalar}_+\rangle$. We determine the energy levels and the wave functions and also discuss their stability.

Keywords: Solitons Monopoles and Instantons, Supersymmetry and Duality.

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1. Introduction

The presence of a 't Hooft-Polyakov monopole [1, 2] gives rise to remarkable effects such as baryon number violation [3, 4]. It is also believed that there must exist processes where a charged particle transfers charge to the monopole, converting it into a dyon (see e.g. [5]–[8]), or more generally, that there are charge transfer processes in the interaction of particles with dyons. These effects are believed to be quite generic, peculiar to the nature of the monopole field, but otherwise model independent. In particular, they are present in supersymmetric models.

Here we will consider specifically the Prasad-Sommerfield dyon solution of $N = 2$ supersymmetric SU(2) Yang-Mills theory [9] (for reviews see [10, 11]) and its interaction with scalar particles of a SU(2) multiplet in the adjoint and in the fundamental representation.

In general, the equations of the scalar field in the background of the dyon couple the different members of the SU(2) multiplet. By defining the electric charge in terms of the asymptotic states where the Higgs field is constant, far from the dyon core, this fact is seen as a nonconservation of the charge of the scalar field. Since it is possible to define the *total* electric charge by the flux at infinity of the total electric field (that is, including the electric field of the dyon solution) and show that it is conserved, this implies that in the process there is a transfer of charge from the particle to the dyon core. While it is difficult to incorporate exactly the back reaction of the scalar field on the dyon solution, it is possible to take into account the main effect of the total charge conservation by representing the charge degrees of freedom of the dyon by means of a quantum rotator formalism ([12, 11], see in particular ref. [6]). The outcome is that in general the quantum states of the particle-dyon system are of the form of quantum entanglement, i.e. a linear superposition of particle-dyon states in which the total charge is fixed, but the particle and the dyon appear with various charge assignements.

It is interesting to compare with the string theory description of the $N = 2$ monopole in terms of D branes. The $N = 2$ SU(2) monopole can be geometrically described as a tube (representing a D string) connecting two parallel D3 branes [13]. The vacuum expectation value of the Higgs field is proportional to the distance between the D3 branes. This separation breaks the SU(2) gauge group to U(1). Consider a small open string representing a neutral particle on one of the two D3-branes: one of the string end points, say the negatively charged one, can fall into the monopole and pass through it to the other brane, whereas the other endpoint may remain on the first brane. An observer on the first brane would see a positively charged particle and a negative charge dyon, due to the flux lines of the electric field dragged by the particle which falled into the monopole. The string would go from the positive particle to the dyon (providing a physical realization of the mathematical string introduced in ref. [8] for preserving gauge invariance) and pass through the tube up to the negative end point on the other brane. Clearly, it is equally probable that the positively charged endpoint of the original open string falls into the monopole, which would lead to a system of a positive charge dyon and a negative charge particle on the first brane. Of course this geometrical picture is unable to take into account the quantum effect of entanglement, whereby the resulting state can be a superposition of the two possible realities: a dyon $(+)$ and a particle $(-)$, or a dyon $(-)$ and a particle $(+)$.

In this paper we consider the possible bound states of a dyon and a charged scalar particle. We first perform a general harmonic analysis in the case of the finite-size monopole and, after diagonalizing the angular momentum, we get coupled radial equations, where the off-diagonal terms — representing the coupling of different charges — vanish exponentially outside the dyon core. As a result, we find that the possible bound states are always of the entangled form. In other words, a bound state just of the form, say, $|\text{dyon}_-\rangle|\text{scalar}_+\rangle$ (a plus charged scalar particle and a minus charged dyon) can never occur. Rather, the possible bound states are a linear combination of the above state with the state of opposite charge, $|\text{dyon}_+\rangle|\text{scalar}_-\rangle$. At large distances, the system is similar to a hydrogen atom, with a particle orbiting far away from the core. We will find that the energy spectrum and wave functions can indeed be approximated by the same formulas of the hydrogen atom, with a proper definition of the effective charge and effective angular momentum. The most relevant effect of the core is thus the production of quantum entanglement.

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Bound states of particles with dyons have been studied in the past (see e.g. [14, 15]), mainly for fermions and in the point-like core limit, except for the special zero energy state found by Jackiw and Rebbi [16]. However, the peculiar fact that bound states to dyons are always quantum entangled with the system of opposite charges does not emerge in this limit. In ref. [17, 18] a more refined analysis was performed, by keeping into account the core interior in some stepwise approximation, but focusing on the case of the monopole or for a fixed dyon, and mainly for investigations on the Jackiw-Rebbi phenomenon; dynamical effects of charge transfer and related collective dyon modes were not taken into account.

In order to observe the quantum entanglement effect, the key points are: *a*) the interactions in the monopole interior (in particular, the effect would not appear in a large distance approximation neglecting what happens inside the core); *b*) the account of the dyon degrees of freedom and of the collective mode which is responsible for charge conservation. Here we report on this phenomenon, not only by controlled analytic approximations but also providing the exact numerical solutions to the bound state problem.

Our computation is done for the case of globally neutral systems of scalars and dyons, for scalars in the adjoint and fundamental representation of SU(2), for the case of scalars belonging to the $N = 2$ supermultiplet and also more in general. We also discuss briefly the stability of the bound states against perturbations not included in our computation, in particular, the radiation of e.m. quanta. A summary of the results is reported in the last section 5.

2. Dyons in $N = 2$ SU(2) super Yang-Mills

2.1 Monopole and dyon solutions

Let σ^a, $a = 1, 2, 3$ be the Pauli matrices, satisfying $[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c$. The background describing the dyon of $N = 2$ SU(2) super Yang-Mills theory is given by [9, 10]

$$A^i = A^i_b\sigma^b = -\sigma_b\epsilon_{bij}\frac{r^j}{er^2}\, f(x)\,, \qquad \Phi_b\sigma^b = \frac{\vec{r}.\vec{\sigma}}{r}aG(x)\,, \tag{2.1}$$

$$A^0 = \frac{\vec{r}.\vec{\sigma}}{2r}aG(x)\sin\Theta\,, \qquad x \equiv \mathrm{ear}\cos\Theta\,, \tag{2.2}$$

$$f(x) = -\frac{1}{2}(1 - K(x))\,, \qquad K(x) = \frac{x}{\sinh(x)}\,, \qquad G(x) = \coth x - \frac{1}{x}\,, \tag{2.3}$$

where a represents the expectation value of a scalar field, and $(ea\cos\Theta)^{-1}$ determines the dyon size. The behavior at large x and small x is

$$K(x) = O(e^{-x})\,, \qquad G(x) = 1 - \frac{1}{x} + O(e^{-x})\,, \qquad x \gg 1\,, \tag{2.4}$$

$$K(x) = 1 - \frac{x^2}{6} + O(x^4)\,, \qquad G(x) = \frac{x}{3} + O(x^3)\,, \qquad x \ll 1\,. \tag{2.5}$$

The dyon charge and mass are given by

$$\begin{aligned} q &= g\tan\Theta = -\frac{4\pi}{e}\tan\Theta\,, \qquad q = ne\,, \\ M &= a\sqrt{g^2 + n^2e^2} = \left|\frac{ag}{\cos\Theta}\right|\,. \end{aligned} \tag{2.6}$$

For $e^2 \ll 1$, this becomes

$$M \cong |ag| \left(1 + \frac{1}{2}\alpha^2\right), \qquad \alpha \ll 1\,, \tag{2.7}$$

$$\alpha = \tan\Theta = -\frac{eq}{4\pi}\,. \tag{2.8}$$

In particular, the mass difference between a dyon and a monopole is given by

$$\Delta E \equiv M_{\rm dyon} - M_{\rm mon} \cong \frac{1}{2}|aq\alpha|\,. \tag{2.9}$$

The Coulomb potential has a constant piece at infinity, related to the energy associated with the dyon electric field. For the analysis in section 3, it is convenient to write it as

$$A^0 = \frac{\vec{r}.\vec{\sigma}}{2r} V \hat{q}\,, \qquad V = -\frac{ea}{4\pi} G(x) \cos\Theta\,. \tag{2.10}$$

where $\hat{q}$ represents the dyon charge operator.

2.2 Charge conservation

Let $\Phi = \Phi_b \sigma^b$ be the Higgs field which has a constant expectation value at infinity, and let Ψ_n stand for the other scalar fields. We choose the gauge $A_0 = 0$.

The lagrangian is invariant under the transformation

$$\delta\Phi = 0\,, \qquad \delta\Psi_n = i[\epsilon\Phi, \Psi_n]\,, \qquad \delta A_l = \epsilon\partial_l\Phi + i[\epsilon\Phi, A_l]\,.$$

Consider now $\epsilon = \epsilon(t)$ as an arbitrary function of time. The lagrangian is no longer invariant but the variation of the Action with respect to ϵ must be zero because of the equation of motions. We thus find a conserved quantity Q:

$$Q = \int d^3r\, \mathrm{Tr}\left(D_l\Phi\partial_t A_l + i\sum_n \Phi[\Psi_n, \partial_t\Psi_n]\right). \tag{2.11}$$

One can indeed independently check that $\partial_t Q = 0$ by using the field equations. Note that Q is given by an integral of the sum of two densities, the "charge density of the scalar fields" Ψ_n, that is $i\,\mathrm{Tr}\sum_n \Phi[\Psi_n, \partial_t\Psi_n]$ plus the "charge density of the dyon", that is $\mathrm{Tr}(D_l\Phi\partial_t A_l)$. A similar proof for charge conservation is given in [6] for the case of SU(2) Yang-Mills theory with an isodoublet of Dirac fermions.

Using the field equation

$$D_l\partial_t A_l = i\sum_n [\Psi_n, \partial_t\Psi_n] + i[\Phi, \partial_t\Phi]\,,$$

the charge can be written as

$$Q = \int_\Sigma d\vec{\Sigma}\, \mathrm{Tr}[\Phi\partial_t\vec{A}]\,,$$

where Σ is a closed surface at infinity.

Thus Q is interpreted as the total charge (apart from a constant factor, equal to the expectation value of the Higgs field at infinity) since it is measured by the total electric flux in the spontaneously broken theory. Quantum mechanically, if the initial state is an eigenstate of Q, also the final state must be an eigenstate of Q, since Q commutes with the hamiltonian.

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3. Scalar particles in the adjoint representation

3.1 Covariant equations

Let us consider a scalar particle in the dyon background. The scalar particle is a quantum of the matrix valued scalar field Ψ: we take an SU(2) triplet $\Psi = \Psi_a \sigma^\alpha$. The equation of motion is given by

$$D_\mu^2 \cdot \Psi = \frac{1}{4} e^2 [\Phi, [\Phi, \Psi]] \,, \tag{3.1}$$

where $D_\mu^2 \cdot \Psi \equiv D_\mu \circ (D_\mu \circ \Psi)$, $D_\mu \circ \Psi \equiv \partial_\mu \Psi - ie[A_\mu, \Psi]$.

We look for solutions of eq. (3.1) of the form: $\Psi(\vec{r}, t) = e^{-iEt}\psi(\vec{r})$, and $\psi(\vec{r})$ is interpreted as the stationary wave function of the scalar particle.

It is convenient to choose the following basis for SU(2) Lie algebra:

$$\begin{aligned} \hat{\alpha} &= u_i \sigma_i \,, & \hat{\beta} &= v_i \sigma_i \,, & \hat{\gamma} &= n_i \sigma_i \,, \\ u_i &= \delta_{3i} - n_3 n_i \,, & v_i &= \epsilon_{3ji} n_j \,, & n_i &= \frac{r_i}{r} \,. \end{aligned} \tag{3.2}$$

Note that $\vec{u}/n_T$, $\vec{v}/n_T$, $\vec{n}$, with $n_T^2 \equiv 1 - n_3^2$ represent an orthonormal frame for vectors in $\mathbb{R}^3$, $\vec{u}.\vec{v} = \vec{u}.\vec{n} = \vec{n}.\vec{v} = 0$, $\vec{n}^2 = 1$, $\vec{u}^2 = \vec{v}^2 = n_T^2$. It follows that $\mathrm{Tr}\,\hat{\alpha}\hat{\beta} = \mathrm{Tr}\,\hat{\alpha}\hat{\gamma} = \mathrm{Tr}\,\hat{\gamma}\hat{\beta} = 0$, $\frac{1}{2}\mathrm{Tr}\,\hat{\alpha}^2 = \frac{1}{2}\mathrm{Tr}\,\hat{\beta}^2 = n_T^2$, $\frac{1}{2}\mathrm{Tr}\,\hat{\gamma}^2 = 1$.

The matrices $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ obey the commutation relations:

$$[\hat{\alpha}, \hat{\gamma}] = 2i\hat{\beta} \,, \qquad [\hat{\beta}, \hat{\gamma}] = -2i\hat{\alpha} \,, \qquad [\hat{\alpha}, \hat{\beta}] = -2in_T^2 \hat{\gamma} \,,$$

and

$$[\hat{\alpha}_\pm, \hat{\gamma}] = \pm 2\hat{\alpha}_\pm \,, \qquad \hat{\alpha}_\pm = \hat{\alpha} \pm i\hat{\beta} \,.$$

The matrix valued wave function $\psi(\vec{r})$ can be decomposed as

$$\psi(\vec{r}) = F_+(\vec{r})\,\hat{\alpha}_- + F_-(\vec{r})\,\hat{\alpha}_+ + F_0(\vec{r})\hat{\gamma} \,. \tag{3.3}$$

The component $F_0(\vec{r})$ multiplying $\hat{\gamma}$ represents a neutral component, whereas the components $F_\pm$ of $\hat{\alpha} \mp i\hat{\beta}$ represent charge e and charge $-e$ components, respectively. This is clear by a gauge transformation so that

$$U^{-1}\hat{\gamma}\,U = \sigma_3 \,, \qquad U^{-1}(\hat{\alpha} \pm i\hat{\beta})U = -2\sin\theta\, e^{\pm i\varphi}\sigma_\mp \,, \tag{3.4}$$

where

$$U(\vec{n}) = \cos\frac{\theta}{2} + i \sin\frac{\theta}{2}\,\frac{\epsilon_{3ij}\sigma_i n_j}{\sin\theta} \,. \tag{3.5}$$

In this gauge, the charge operator is $\hat{Q} = \frac{e}{2}\sigma_3$, acting on a state Ψ by $\hat{Q} \cdot \Psi \equiv [\hat{Q}, \Psi]$.

Now we evaluate $\vec{D}^2 \cdot \psi$. With the definitions (1.4), and further defining for short $\Omega \equiv \vec{\partial}^2 - \frac{1}{r^2}(1 + K^2) - \frac{2}{r}\frac{n_3}{n_T^2}u_j\partial_j$, we find:

$$\vec{D}^2 \cdot (\hat{\gamma}F_0(\vec{r})) = \hat{\gamma}\left(\vec{\partial}^2 - \frac{2K^2}{r^2}\right)F_0 + \hat{\alpha}\left(\frac{2K}{rn_T^2}u_j\partial_j\right)F_0 + \hat{\beta}\left(\frac{2K}{rn_T^2}v_j\partial_j\right)F_0 \,. \tag{3.6}$$

$$\vec{D}^2 \cdot [(\hat{\alpha} \pm i\hat{\beta})F_\mp] = (\hat{\alpha} \pm i\hat{\beta})\left(\Omega \pm \frac{2}{r}\frac{n_3}{n_T^2}iv_j\partial_j\right)F_\mp + \hat{\gamma}\frac{2K}{r}\left(\mp iv_j\partial_j - u_j\partial_j + \frac{2n_3}{r}\right)F_\mp \,. \tag{3.7}$$

3.2 Harmonic analysis

To solve for the angular dependence, it is convenient to use spherical coordinates r, θ, φ. We have:

$$u_j \partial_j = -\frac{1}{r} \sin\theta \partial_\theta \,, \qquad i v_j \partial_j = i \frac{1}{r} \partial_\varphi \,. \tag{3.8}$$

Asymptotically at $r \to \infty$, we have $K \to 0$ and in this case there is decoupling of charges $(+, -, 0)$ in (3.6) and (3.7), as expected. Consider the monopole case $\Theta = 0$. Charged particles have mass m, with $m = ea$. Define

$$F_\pm(\vec{r}) = \frac{e^{\pm i\varphi}}{\sin\theta} h_\pm(\vec{r}) \,. \tag{3.9}$$

We find that for $r \gg m^{-1}$ eqs. (3.1), (3.7) give

$$\begin{aligned} \Bigg(\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2\sin^2\theta}(\partial_\varphi \mp i(1-\cos\theta))^2 + \\ + E^2 - (m - \frac{1}{r})^2 \Bigg) h_\mp(\vec{r}) = 0 \,. \end{aligned}$$

This is the covariant equation for a scalar particle of charge $\mp e$ moving in a U(1) point-like Dirac monopole background. In addition, there is a Coulomb-like potential due to the long range tail of the Higgs field.

Let us now consider the complete equations in all the space. In the Appendix we report the harmonic analysis in full detail. The angular dependence is solved by setting

$$\begin{aligned} F_0(\vec{r}) &= \phi_0(r) Y_{lm}(\theta,\varphi) \,, \\ h_+(\vec{r}) &= \frac{1}{l_0}\phi_+(r) Z^+_{lm}(\theta,\varphi) \,, \qquad h_-(\vec{r}) = \frac{1}{l_0}\phi_-(r) Z^-_{lm}(\theta,\varphi) \,, \\ l_0 &\equiv \sqrt{l(l+1)} \,, \end{aligned} \tag{3.10}$$

where $Y_{lm}(\theta,\varphi)$ are the standard spherical harmonics, and $Z^\pm_{lm}$ can be expressed in terms of them:

$$Z^\pm_{lm} = \frac{e^{\mp i\varphi}}{\sin\theta}(\sin\theta\partial_\theta \mp i\partial_\varphi) Y_{lm} \,. \tag{3.11}$$

Note that the angular expansion of the charged components $h_\pm$ begins with $l = 1$.

Using the results of the appendix, we get the following system of coupled differential equations for $\phi_0(r)$, $\phi_\pm(r)$:

$$\left(-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{2K^2}{r^2} + \frac{1}{r^2} l(l+1) \right)\phi_0 + \frac{2Kl_0}{r^2}(\phi_+ + \phi_-) = E^2 \phi_0 \,, \tag{3.12}$$

$$\left[-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2}(-1+K^2) + \frac{1}{r^2} l(l+1) \right]\phi_+ + \frac{Kl_0}{r^2}\phi_0 = \left[(E - eV\hat{q})^2 - e^2a^2G^2\right]\phi_+ \,, \tag{3.13}$$

$$\left[-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2}(-1+K^2) + \frac{1}{r^2} l(l+1) \right]\phi_- + \frac{Kl_0}{r^2}\phi_0 = [(E + eV\hat{q})^2 - e^2a^2G^2]\phi_- \,, \tag{3.14}$$

where V was defined in eq. (2.10).

3.3 Collective coordinates and equations of motion

When a process of charge transfer occurs, there is an important back reaction in the dyon background. This effect can be taken into account by incorporating into the dynamics the collective coordinate χ associated to global U(1) gauge transformations. This "quantum rotator" degree of freedom ensures charge conservation. The dyon charge operator is written as $\hat{q} = g\tan\Theta = -ie\partial_\chi$.

Consider a system consisting of a scalar particle and a dyon with total charge equal to zero. The wave function can be written in the form

$$\psi = \begin{pmatrix} \phi_+ e^{-i\chi} \\ \phi_0 \\ \phi_- e^{i\chi} \end{pmatrix} . \tag{3.15}$$

In general, this represents a mixture of monopole and neutral scalar, with dyons and scalars of charges $(+-)$ and $(-+)$. When the system makes a transition from a monopole to a dyon, the field component A_0 is turned on. In addition, the energy of the scalar is reduced into an amount equivalent to the mass difference between the dyon and the monopole. In what follows we will assume that $\alpha = e^2/(4\pi)$ is small. If α is not small, other back reaction effects become important and this semiclassical analysis is not applicable (for $\alpha \ll 1$, one may approximate $\sin\Theta \cong \tan\Theta$, $\cos\Theta = 1 + O(e^4) \cong 1$).

The equations of motion (3.12)–(3.14) can be written as

$$H\psi = 0\,, \tag{3.16}$$

where

$$\begin{aligned} H &= \left(\hat{P} + (E + msGi\partial_\chi)^2 - m^2G^2\right)\mathbf{1}_{3\times 3} + \frac{l_0}{r^2}\hat{M}\,, \\ \hat{M} &= \begin{pmatrix} 0 & Ke^{-i\chi} & 0 \\ Ke^{i\chi} & 0 & Ke^{-i\chi} \\ 0 & Ke^{i\chi} & 0 \end{pmatrix}, \qquad s \equiv \sin\Theta\,, \\ \hat{P} &= -\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2}l(l+1) + \frac{2K^2}{r^2} + \frac{1}{r^2}(1+K^2)\partial_\chi^2\,. \end{aligned} \tag{3.17}$$

Then the equations of motion for the neutral system take the form

$$\left(-\frac{1}{r^2}\partial_r r^2\partial_r + \frac{2K^2}{r^2} + \frac{1}{r^2}l(l+1)\right)\phi_0 + \frac{2Kl_0}{r^2}(\phi_+ + \phi_-) = E^2\phi_0\,, \tag{3.18}$$

$$\left[-\frac{1}{r^2}\partial_r r^2\partial_r + \frac{1}{r^2}(-1+K^2) + \frac{1}{r^2}l(l+1)\right]\phi_+ + \frac{Kl_0}{r^2}\phi_0 = [(E - msG)^2 - m^2G^2]\phi_+\,, \tag{3.19}$$

$$\left[-\frac{1}{r^2}\partial_r r^2\partial_r + \frac{1}{r^2}(-1+K^2) + \frac{1}{r^2}l(l+1)\right]\phi_- + \frac{Kl_0}{r^2}\phi_0 = [(E - msG)^2 - m^2G^2]\phi_-\,. \tag{3.20}$$

The incorporation of the collective coordinate χ produces the flip of sign of the dyon charge in the equations for ϕ_+ and ϕ_-, required by charge conservation. As a result, the Coulomb potential of the dyon-scalar is always attractive for both ϕ_+ and ϕ_-.

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In terms of the variables $\phi_{\rm a} = \phi_+ - \phi_-$ and $\phi_{\rm s} = \phi_+ + \phi_-$, we have a decoupled equation

$$\left[-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2}(-1+K^2) + \frac{1}{r^2}l(l+1)\right]\phi_{\rm a} = [(E - msG)^2 - m^2G^2]\phi_{\rm a}\,, \tag{3.21}$$

and a coupled system of ordinary differential equations for $\phi_{\rm s}$ and ϕ_0,

$$\left[-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2}(-1+K^2) + \frac{1}{r^2}l(l+1)\right]\phi_{\rm s} + \frac{2Kl_0}{r^2}\phi_0 = [(E - msG)^2 - m^2G^2]\phi_{\rm s}\,, \tag{3.22}$$

$$\left(-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{2K^2}{r^2} + \frac{1}{r^2}l(l+1)\right)\phi_0 + \frac{2Kl_0}{r^2}\phi_{\rm s} = E^2\phi_0\,. \tag{3.23}$$

In the section 3.4, we will see that eq. (3.21) describes bound states. The wave function for such bound state is of the form $|\psi\rangle = |{\rm dyon}_+\rangle|{\rm scalar}_-\rangle - |{\rm dyon}_-\rangle|{\rm scalar}_+\rangle$, i.e. there is quantum entanglement of charges with a dyon state. Eqs. (3.22), (3.23) can be used to describe the dynamics of a scattering process in which a neutral scalar particle scatters off a monopole and gives rise to an outgoing state which is a mixture of a neutral scalar and a $+/-$ charged scalar entangled with a $-/+$ charged dyon.

3.4 Bound states

Here we study the solutions of the equation (3.21) for $\phi_{\rm a}$. In terms of the radial coordinate $x = (m\cos\Theta)r$, it takes the form

$$\left[-\frac{1}{x^2}\partial_x x^2 \partial_x + \frac{1}{x^2}(-1+K^2(x)) + \frac{1}{x^2}l(l+1)\right]\phi_{\rm a} = [\mathcal{E}^2 - 2\mathcal{E}\alpha G(x) - G^2(x)]\phi_{\rm a}\,, \tag{3.24}$$

$$\mathcal{E} \equiv \frac{E}{m\cos\theta}\,, \qquad \alpha = \tan\Theta = \frac{e^2}{4\pi}\,. \tag{3.25}$$

This is a Schrödinger-type equation with a potential which is an attractive Coulomb potential at large distances, where $G \cong 1 - 1/x$ (see eq. (2.4)), and therefore it may admit bound state solutions.

A closed analytic solution of this differential equation is not known. We will solve it by using two independent methods: *a*) Analytic method; *b*) Numerical method.

Let us begin with the analytic method. We approximate the potential by a simpler function, replacing K^2 and G by K_0^2, G_0 defined as follows (cf. eqs. (2.4), (2.5)):

$$\begin{aligned} K_0^2(x) &= 0\,, \qquad G_0(x) = 1 - \frac{1}{x}\,, \qquad x > 1\,, \\ K_0^2(x) &= 1 - \frac{x^2}{3}\,, \qquad G_0(x) = 0\,, \qquad x \le 1\,. \end{aligned} \tag{3.26}$$

Exterior solutions: For $x > 1$ the equation is

$$\left[-\frac{1}{x^2}\partial_x x^2 \partial_x + \frac{l(l+1)}{x^2} - \frac{2}{x}(\alpha\mathcal{E} + 1) + k^2\right]\phi_{\rm a} = 0 \tag{3.27}$$

with

$$k = \sqrt{1 + 2\alpha\mathcal{E} - \mathcal{E}^2}\,. \tag{3.28}$$

Note that the Coulomb potential in eq. (3.27) has two contributions, one coming from the interaction with the Higgs field, the other from A_0. For $\alpha \ll 1$, the Coulomb potential due to the Higgs field is dominant.

Solutions with real k represent bound states. The solution which vanishes at infinity is the confluent hypergeometric function:

$$\phi_{\mathrm{a}}^{ex} = x^l e^{-kx}\Psi(a, 2+2l, -2kx)\,, \qquad a = 1 + l - \frac{\alpha\mathcal{E}+1}{k}\,. \tag{3.29}$$

Equation (3.27) is formally the same as the Schrödinger equation for the Hydrogen atom. Thus, for bound states with wavefunctions having support at $x \gg 1$, the energy eigenvalues are approximately determined by the hydrogen atom formula. This formula follows by demanding that the wave function be regular at the origin, which amounts to say that the parameter a is an integer ≤ 0. Thus in this case the eigenvalues are given by

$$n = n_0 + l + 1 = \frac{\alpha\mathcal{E}+1}{k}\,, \qquad n_0 = 0, 1, 2, \dots \tag{3.30}$$

or

$$\mathcal{E} = \frac{\alpha(n^2-1) \pm n\sqrt{n^2-1}\sqrt{\alpha^2+1}}{n^2+\alpha^2}\,, \tag{3.31}$$

$$\cong \pm\sqrt{1-\frac{1}{n^2}} + \alpha\left(1-\frac{1}{n^2}\right), \qquad \alpha \ll 1\,. \tag{3.32}$$

We shall see that (3.32) is a very good approximation for all eigenstates with $l > 1$.

Note that there are also eigenvalues of negative energies. The negative energy eigenstates should be interpreted in terms of the antiparticle of opposite charge (thus $\alpha \to -\alpha$, see eq. (2.8)) and positive energy. Reversing the sign of the energy has the same effect in the equation as reversing the sign of α. These states no longer correspond to the neutral system under discussion, but to a system of total electric charge equal to $2e$ (or $-2e$). They are bound to the dyon because the Coulomb potential is still attractive thanks to the Higgs contribution (i.e. $\alpha\mathcal{E} < 0$ but $1 + \alpha\mathcal{E} > 0$). Here we will discuss only eigenvalues of positive energies.

Note that there are bound states even in the limit $\alpha \to 0$. The origin of the binding force in this case is the Higgs field.

The binding energy is represented by

$$k^2 = 1 + 2\alpha\mathcal{E} - \mathcal{E}^2 = (\mathcal{E}_\infty^+ - \mathcal{E})(\mathcal{E} - \mathcal{E}_\infty^-)\,,$$

with

$$\mathcal{E}_\infty^\pm = \alpha \pm \sqrt{1+\alpha^2}\,.$$

Here $\mathcal{E}_\infty^+$ represents the asymptotic mass.

Interior solutions: For $x < 1$ the equations are

$$\left[-\frac{1}{x^2}\partial_x x^2 \partial_x + \frac{l(l+1)}{x^2} - k_{\mathrm{in}}^2\right]\phi_{\mathrm{a}} = 0\,, \tag{3.33}$$

l	n_0	$k_{\rm num}$	$k_{\rm an}$	$k_{\rm point}$
1	0	0.5724	0.5953	0.6023
1	1	0.3937	0.4048	0.4089
2	0	0.4076	0.4094	0.4089
2	1	0.3076	0.3086	0.3086
3	0	0.3086	0.3086	0.3086
3	1	0.2476	0.2476	0.2476

Table 1: Eigenvalues for binding energy for $\alpha = 0.2$. $k_{\rm num}$ are the eigenvalues obtained by numerical integration while $k_{\rm an}$ are obtained by the analytic method. The last column $k_{\rm point}$ are the eigenvalues in the point-like limit, where the differential equation is formally the same as the hydrogen atom equation (using (3.32)).

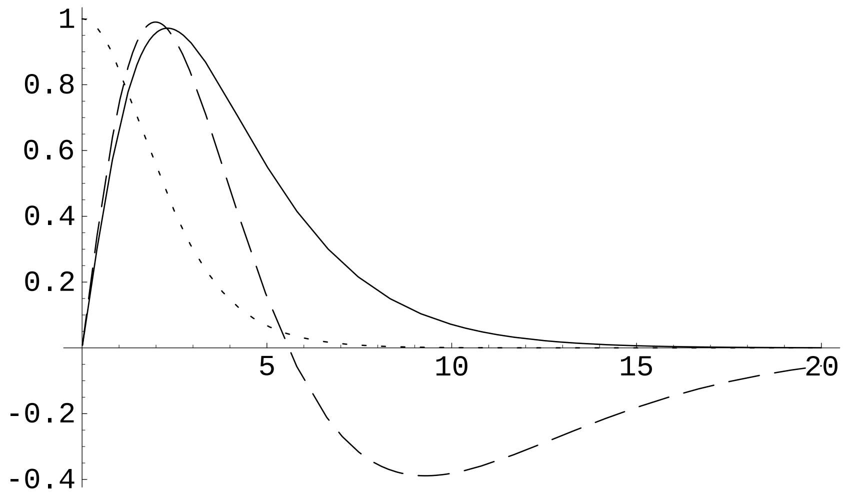

Figure 1: Wave functions for the ground state (solid line) and first excited state (dashed line) with $l = 1$, $n_0 = 0, 1$, respectively, for $\alpha = 0.2$. The dotted line is the plot of $K^2(x)$, which indicates the region of the dyon core.

with

$$k_{\rm in}^2 = \mathcal{E}^2 + \frac{1}{3}\,.$$

The solution that is regular at $r = 0$ is given by

$$\phi_{\rm a}^{in} = c_0 J_{l+\frac{1}{2}}(k_{\rm in} x)\,. \tag{3.34}$$

The energy eigenvalues follow from imposing continuity of the wavefunction and its first derivative at $x = 1$. This gives the condition

$$\left(\partial_x \phi_{\rm a}^{in}\, \phi_{\rm a}^{ex} - \phi_{\rm a}^{in}\, \partial_x \phi_{\rm a}^{ex}\right)\Big|_{x=1} = 0\,. \tag{3.35}$$

The resulting eigenvalues (in terms of $k^2 = 1 + 2\alpha\mathcal{E} - \mathcal{E}^2$) are given in table 1.

The numerical determination of the eigenvalues is performed by integrating the differential equation from $x = 0$ to some $x = x_{\rm max} \gg 1$, with regular boundary conditions at zero. The energy eigenvalues are then determined by the condition that the wavefunction approaches zero at $x = x_{\rm max}$ (equivalently, that the wave function approaches the asymptotic form (3.29)). The results are given in table 1. Figure 1 is a plot of the wave functions for the ground state and the first excited level.

A final remark about the stability of the bound states. The theory contains also photons, i.e. the quanta of the (far from the core) unbroken U(1) gauge field, which are coupled to our charged scalar particles. The coupling is proportional to the charge operator, which has nonvanishing matrix elements between the states $\phi_{\rm a}$ and the states $\phi_{\rm s}$. Because $\phi_{\rm s}$ is coupled to the massless scalar ϕ_0, this implies an instability of the bound state. The dominant decay channel of the fundamental state $\phi_{\rm a}$ is by single photon emission, and subsequently conversion of $\phi_{\rm s}$ into a ϕ_0 particle that can get to infinity. A further discussion of this is given in section 5.

4. Scalar particles in the fundamental representation

4.1 Covariant equations

We will work in the gauge where the charge operator is $\hat{Q} = \frac{e}{2}\sigma_3$. Now the wave function $\psi(\vec{r})$ is an isospinor, that is an isospin doublet, and the charge operator acts on it in the standard way as a 2×2 matrix on an isospinor. In this gauge, the space components of the covariant derivatives are

$$\begin{aligned}\hat{D}_l &\equiv U^{-1}(\partial_l - ieA_l)U \\ &= \partial_l + \frac{i}{2r(1-n_3^2)}\left[K(u_l\vec{v}\cdot\vec{\sigma} + v_l\vec{n}_T\cdot\sigma) + v_l(1-n_3)\sigma_3)\right].\end{aligned} \tag{4.1}$$

Here we use the notation of section 3.1, with U defined in eq. (3.5).

Notice that for $K \cong 0$ (i.e., far away from the core) one has

$$\begin{aligned}\hat{D}_l^2 &= \left(\partial_l + \frac{iv_l}{2r(1+n_3)}\sigma_3\right)^2 \\ &= \frac{1}{r^2}\partial_r r^2\partial_r + \frac{1}{r^2\sin^2\theta}\left(\sin\theta\partial_\theta(\sin\theta\partial_\theta) + \left(\partial_\varphi + \frac{i}{2}(1-\cos\theta)\sigma_3\right)^2\right).\end{aligned}$$

This is the covariant Laplace operator for a scalar particle of charge $\pm\frac{1}{2}e$ moving in a U(1) Dirac monopole background.

The full equations can be written as:

$$-\hat{D}_l^2\cdot\psi = \left(-\frac{1}{r^2}\partial_r r^2\partial_r + \frac{K^2}{2r^2} - \frac{1}{4r^2} + \frac{1}{r^2}\hat{J}_{1/2}^2\right)\psi + \frac{K}{r^2}M\cdot\psi\,, \tag{4.2}$$

where

$$\begin{aligned}\hat{J}_{1/2}^2 &= \begin{pmatrix}\hat{J}_{+1/2}^2 & 0 \\ 0 & \hat{J}_{-1/2}^2\end{pmatrix}, \qquad M = \frac{1}{\sin\theta}\begin{pmatrix}0 & e^{-i\varphi}\nabla_{-1/2} \\ -e^{i\varphi}\bar{\nabla}_{1/2} & 0\end{pmatrix}, \\ \psi &= \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}.\end{aligned} \tag{4.3}$$

Now

$$\hat{J}^2_{\pm 1/2} = -\frac{1}{\sin^2\theta}\nabla_{\pm 1/2}\bar{\nabla}_{\pm 1/2} \tag{4.4}$$

with

$$\begin{aligned}\nabla_{\pm 1/2} &= \sin\theta\partial_\theta \mp i\partial_\varphi + \frac{1}{2}(1-\cos\theta)\,,\\ \bar{\nabla}_{\pm 1/2} &= \sin\theta\partial_\theta \pm i\partial_\varphi - \frac{1}{2}(1-\cos\theta)\,.\end{aligned} \tag{4.5}$$

4.2 Harmonic analysis

Let $Z^{(\pm 1/2)}_{lm}$ be an eigenfunction of the operator

$$\tilde{L}^2_{\pm} = \hat{J}^2_{\pm 1/2} - \frac{1}{2} + \frac{1}{4}\,, \tag{4.6}$$

In particular consider

$$\tilde{L}^2_{+} Z^{(1/2)}_{lm} = l(l+1) Z^{(1/2)}_{lm}\,.$$

Explicit expressions for these eigenfunctions are given in ref. [19, 20]. The eigenvalues are now $l = \frac{1}{2}, \frac{1}{2}+1, \frac{1}{2}+2, \ldots$ Note that $\hat{J}^2_{\pm 1/2} = (l+1/2)^2$. It is convent to fix a phase convention by defining

$$Z^{(-1/2)}_{lm} \equiv -\frac{1}{\mu}\frac{e^{i\varphi}}{\sin\theta}\bar{\nabla}_{1/2} Z^{(1/2)}_{lm}\,, \qquad \mu \equiv l + \frac{1}{2}\,. \tag{4.7}$$

It is indeed easy to verify that

$$Z^{(1/2)}_{lm} = \frac{1}{\mu}\frac{e^{-i\varphi}}{\sin\theta}\nabla_{-1/2} Z^{(-1/2)}_{lm}\,, \qquad \tilde{L}^2_{-} Z^{(-1/2)}_{lm} = l(l+1) Z^{(-1/2)}_{lm}\,. \tag{4.8}$$

The angular dependence is then solved by writing

$$\psi_+ = \eta_+(r) Z^{(1/2)}_{lm}(\theta,\varphi)\,, \qquad \psi_- = \eta_-(r) Z^{(-1/2)}_{lm}(\theta,\varphi)\,.$$

Next, we compute $\hat{D}^2_0 \cdot \Psi$, $\hat{D}_0 = U^{-1}(\partial_0 - \frac{1}{2}ieA_0)U$. Using $U^{-1}\gamma U = \sigma_3$, we find

$$\hat{D}^2_0 \cdot \Psi = -\left(E - \frac{1}{2}e\hat{q}V\sigma_3\right)^2 \Psi\,. \tag{4.9}$$

Now we consider a system of vanishing total charge. To incorporate charge conservation, as in section 3.2 we write the dyon charge operator as $\hat{q} = -ie\partial_\chi$, and the wave function as

$$\psi = \begin{pmatrix} \eta_+ e^{-\frac{1}{2}i\chi} \\ \eta_- e^{\frac{1}{2}i\chi} \end{pmatrix}. \tag{4.10}$$

The net effect is that the cross term in the square appearing on the right hand side of (4.9) (which contains the Coulomb interaction) has the same sign for the upper and lower components, i.e. it is an attractive potential for both components.

4.3 Case of the $N = 2$ hypermultiplet

From the point of view of $N = 1$ supersymmetry, the $N = 2$ vector multiplet contains a $N = 1$ vector multiplet and a chiral multiplet Φ. In this section we add to the $N = 2$ pure Yang-Mills theory a hypermultiplet in the fundamental representation of SU(2). The hypermultiplet contains two chiral superfields Q and $\tilde{Q}$, which couple to Φ by a term $W = \tilde{Q}\Phi Q$. The fundamental scalars will get a mass due to the coupling to the Higgs field. To simplify the discussion, here we will not add an independent mass term $M\tilde{Q}Q$. In the next subsection, we will consider scalar particles with an arbitrary mass parameter.

Let us consider a neutral system of dyon and scalars with charges $(\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, \frac{1}{2})$. Taking into account the coupling to the Higgs field, we get the following equations for $\eta_+(r)$ and $\eta_-(r)$ (μ being defined in (4.7)):

$$\left[-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{K^2}{2r^2} + \frac{1}{r^2}\left(\mu^2 - \frac{1}{2}\right)\right]\eta_+ + \frac{\mu K}{r^2}\eta_- = \left[\left(E - \frac{1}{2}msG\right)^2 - \frac{1}{4}m^2G^2\right]\eta_+, \tag{4.11}$$

$$\left[-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{K^2}{2r^2} + \frac{1}{r^2}\left(\mu^2 - \frac{1}{2}\right)\right]\eta_- + \frac{\mu K}{r^2}\eta_+ = \left[\left(E - \frac{1}{2}msG\right)^2 - \frac{1}{4}m^2G^2\right]\eta_-, \tag{4.12}$$

where $s = \sin\Theta$. Let us now define $\eta_{\rm s} = \eta_+ + \eta_-$, $\eta_{\rm a} = \eta_+ - \eta_-$. They satisfy the decoupled equations

$$\left[-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{K^2}{2r^2} + \frac{1}{r^2}\left(\mu^2 - \frac{1}{2}\right)\right]\eta_{\rm s} + \frac{\mu K}{r^2}\eta_{\rm s} = \left[\left(E - \frac{1}{2}msG\right)^2 - \frac{1}{4}m^2G^2\right]\eta_{\rm s}\,, \tag{4.13}$$

$$\left[-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{K^2}{2r^2} + \frac{1}{r^2}\left(\mu^2 - \frac{1}{2}\right)\right]\eta_{\rm a} - \frac{\mu K}{r^2}\eta_{\rm a} = \left[\left(E - \frac{1}{2}msG\right)^2 - \frac{1}{4}m^2G^2\right]\eta_{\rm a}\,. \tag{4.14}$$

Note that the equations for $\eta_{\rm s}$ and $\eta_{\rm a}$ are formally the same under the exchange $\mu \to -\mu$, so they can be investigated on the same footing. In what follows we consider the $\eta_{\rm s}$ equation (4.13), wherefrom we obtain the solutions for $\eta_{\rm a}$ by flipping the sign of μ.

In terms of $x = (m\cos\Theta)r$, the equation (4.13) for $\eta_{\rm s}$ reads

$$\begin{gathered}\left[-\frac{1}{x^2}\partial_x x^2 \partial_x + \frac{K^2(x)}{2x^2} + \frac{1}{x^2}\left(\mu^2 - \frac{1}{2}\right)\right]\eta_{\rm s} + \frac{\mu K(x)}{x^2}\eta_{\rm s} = \left[\mathcal{E}^2 - \alpha\mathcal{E}G(x) - \frac{1}{4}G^2(x)\right]\eta_{\rm s}\,, \\ \mathcal{E} = \frac{E}{m\cos\Theta}\,, \qquad \alpha = \tan\Theta\,. \end{gathered} \tag{4.15}$$

At $x \gg 1$, the differential equation (4.15) takes the form

$$-\frac{1}{x^2}\partial_x x^2 \partial_x \eta_{\rm s} + \left(\frac{l(l+1)}{x^2} - \frac{2\alpha\mathcal{E}+1}{2x} + k^2\right)\eta_{\rm s} = 0\,, \tag{4.16}$$

with

$$k = \sqrt{\frac{1}{4} + \alpha\mathcal{E} - \mathcal{E}^2}\,.$$

The solution which vanishes at infinity is the confluent hypergeometric function:

$$\eta_{\rm s} \cong x^l\, e^{-kx}\Psi(a, 2+2l, -2kx)\,, \tag{4.17}$$

$$a = 1 + l - \frac{2\alpha\mathcal{E}+1}{4k}\,. \tag{4.18}$$

l	n_0	$k_{\rm num}$(s)	$k_{\rm num}$(a)	$k_{\rm point}$
1/2	0	0.1921	0.2245	0.2045
1/2	1	0.1188	0.1310	0.1238
3/2	0	0.1236	0.1241	0.1238
3/2	1	0.08848	0.08881	0.08863
5/2	0	0.08863	0.08863	0.08863
5/2	1	0.069005	0.069005	0.069005

Table 2: Eigenvalues for binding energies $k_{\rm num}$(s,a) for $\eta_{\rm s}$ and $\eta_{\rm a}$ using $\alpha = 0.2$. $k_{\rm point}$ are the approximate eigenvalues given by the analytic expression (4.19).

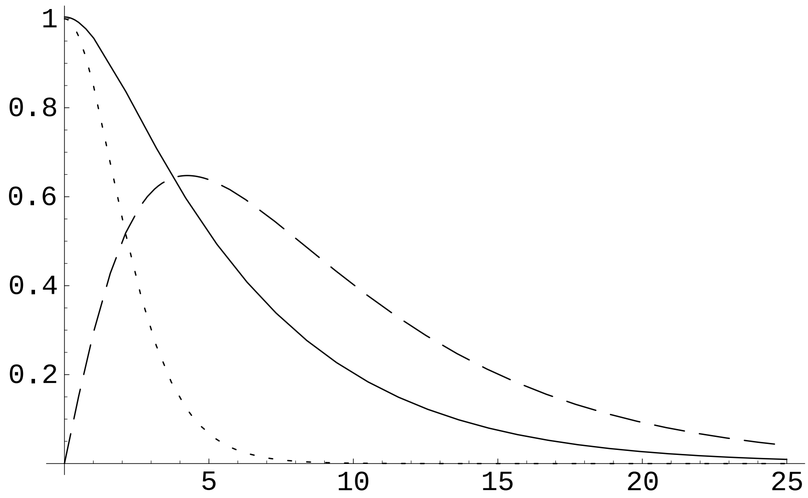

Figure 2: Wave function for the ground state $\eta_{\rm a}$ (solid line) and for $\eta_{\rm s}$ (dashed line) with $l = 1/2$, $n_0 = 0$ (we use $\alpha = 0.2$). The dotted line is a plot of K^2, indicating the region of the dyon core.

The approximate energy eigenvalues are then determined by the condition

$$n_0 + l + 1 = \frac{2\alpha\mathcal{E} + 1}{4\sqrt{\frac{1}{4} + \alpha\mathcal{E} - \mathcal{E}^2}}\,, \qquad n_0 = 0, 1, 2, \ldots \tag{4.19}$$

Some values are given in table 2. Note that energy eigenstates obtained in this approximation are the same for $\eta_{\rm s}$ and $\eta_{\rm a}$, since at $x \gg 1$ $\eta_{\rm a}$ satisfies the same eq. (4.16). This point-like approximation is already good for states with $l = 3/2$ and becomes better for states with high angular momentum, which are farther from the core.

At $x \ll 1$, the equation is

$$-\frac{1}{x^2}\partial_x x^2 \partial_x \eta_{\rm s} + \left(\frac{\mu(\mu+1)}{x^2} - k_{\rm in}^2\right)\eta_{\rm s} = 0\,, \tag{4.20}$$

with

$$k_{\rm in}^2 = \mathcal{E}^2 + \frac{(1+\mu)}{6}\,, \qquad \mu = l + \frac{1}{2}\,.$$

The solutions are Bessel functions.

In the case of $\eta_{\rm a}$, the "centrifugal barrier" for $x \ll 1$ is $\frac{\mu(\mu-1)}{x^2}$. As a result, there is no barrier for the ground state with $\mu = 1$, and the corresponding wave function does not vanish at the origin (see figure 2).

The numerical determination of the eigenvalues is performed as in section 3 by integrating the differential equation from $x = 0$ to some $x = x_{\max} \gg 1$, with regular boundary conditions at zero. The energy eigenvalues are then determined by the normalizability requirement that the wavefunction approaches zero at $x = x_{\max}$. The results are given in table 2.

4.4 Massive scalars in $N \leq 1$ without coupling to Higgs

In the systems of section 3.4 and 4.3, for small α the Coulomb attraction is dominated by the Higgs field. This interaction is dictated by $N = 2$ supersymmetry. As a result, the bound states are not very sensitive to the value of an $\alpha < 1$. It is of interest to investigate systems with less supersymmetry, to see the effect of changing the values of the coupling. Here we will consider a scalar field in the fundamental representation with an arbitrary mass term and no coupling to the Higgs field.

Let us consider a neutral system of dyon and these scalars with charges $(1/2, -1/2)$ and $(-1/2, 1/2)$. In a similar way as in section 4.3, we find the following equations for $\eta_+(r)$ and $\eta_-(r)$:

$$\left[-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{K^2}{2r^2} + \frac{1}{r^2}\left(\mu^2 - \frac{1}{2}\right)\right]\eta_+ + \frac{\mu K}{r^2}\eta_- = \left[\left(E - \frac{1}{2}msG\right)^2 - M^2\right]\eta_+ \,, \quad (4.21)$$

$$\left[-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{K^2}{2r^2} + \frac{1}{r^2}\left(\mu^2 - \frac{1}{2}\right)\right]\eta_- + \frac{\mu K}{r^2}\eta_+ = \left[\left(E - \frac{1}{2}msG\right)^2 - M^2\right]\eta_- \,, \quad (4.22)$$

where we have introduced a mass M. Thus, again the presence of η_+ turns on the other component η_- (and viceversa), so the bound state is an entangled quantum state. The equations are decoupled in terms of $\eta_{\rm s} = \eta_+ + \eta_-$, $\eta_{\rm a} = \eta_+ - \eta_-$, which satisfy

$$\left[-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{K^2}{2r^2} + \frac{1}{r^2}\left(\mu^2 - \frac{1}{2}\right)\right]\eta_{\rm s} + \frac{\mu K}{r^2}\eta_{\rm s} = \left[\left(E - \frac{1}{2}msG\right)^2 - M^2\right]\eta_{\rm s} \,, \quad (4.23)$$

$$\left[-\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{K^2}{2r^2} + \frac{1}{r^2}\left(\mu^2 - \frac{1}{2}\right)\right]\eta_{\rm a} - \frac{\mu K}{r^2}\eta_{\rm a} = \left[\left(E - \frac{1}{2}msG\right)^2 - M^2\right]\eta_{\rm a} \,. \quad (4.24)$$

Introducing the rescaled radial coordinate $x = (m\cos\Theta)r$, the equation for $\eta_{\rm s}$ becomes

$$\left[-\frac{1}{x^2}\partial_x x^2 \partial_x + \frac{K^2(x)}{2x^2} + \frac{1}{x^2}\left(\mu^2 - \frac{1}{2}\right)\right]\eta_{\rm s} + \frac{\mu K(x)}{x^2}\eta_{\rm s} = \left[\left(\mathcal{E} - \frac{1}{2}\alpha G(x)\right)^2 - \mathcal{M}^2\right]\eta_{\rm s} \,,$$
$$\mathcal{E} = \frac{E}{m\cos\Theta}\,, \qquad \mathcal{M} = \frac{M}{m\cos\Theta}\,. \quad (4.25)$$

In the region $x \gg 1$, this differential equation reduces to

$$-\frac{1}{x^2}\partial_x x^2 \partial_x \eta_{\rm s} + \left(\frac{1}{x^2}\left(\mu^2 - \frac{1}{2} - \frac{1}{4}\alpha^2\right) - \frac{\alpha\mathcal{E}}{x} + k^2\right)\eta_{\rm s} = 0\,, \quad (4.26)$$

l	n_0	$k_{\rm num}$(s)	$k_{\rm num}$(a)	$k_{\rm point}$
1/2	0	0.03626	0.03728	0.03666
1/2	1	0.02104	0.02140	0.02117
3/2	0	0.02053	0.02053	0.02053
3/2	1	0.014553	0.014555	0.014545
5/2	0	0.0144460	0.01460	0.014460
5/2	1	0.011217	0.011217	0.011217

Table 3: Eigenvalues for binding energies $k_{\rm num}$(s,a) for $\eta_{\rm s}$ and $\eta_{\rm a}$ using $\alpha = 0.2$ and $\mathcal{M} = 0.5$. $k_{\rm point}$ are the approximate eigenvalues given by the analytic expression (4.28).

with

$$k = \sqrt{\mathcal{M}^2 - \left(\mathcal{E} - \frac{1}{2}\alpha\right)^2} \, .$$

The solution which is normalizable at infinity is the following confluent hypergeometric function:

$$\begin{aligned} \eta_{\rm s} &\cong x^b e^{-kx} \Psi(a, 1+c, -2kx) \, , \\ b &= \frac{1}{2}(-1+c) \, , \qquad a = \frac{1}{2}(1+c) - \frac{\alpha\mathcal{E}}{2k} \, , \qquad c = \sqrt{4\mu^2 - 1 - \alpha^2} \, . \end{aligned} \tag{4.27}$$

The approximate energy eigenvalues in this point-like limit are then determined by the formula

$$-n_0 = \frac{1}{2}(1 + \sqrt{4\mu^2 - 1 - \alpha^2}) - \frac{\alpha\mathcal{E}}{2\sqrt{\mathcal{M}^2 - (\mathcal{E} - \alpha/2)^2}} \, , \qquad n_0 = 0, 1, 2, \ldots \tag{4.28}$$

Some eigenvalues are given in table 3. Note that the binding energies are very small compared to the asymptotic mass $\mathcal{E}_\infty = \mathcal{M} + \frac{1}{2}\alpha$. This can be understood as follows. From eq. (4.28) we see that

$$\Delta\mathcal{E} = \mathcal{E}_\infty - \mathcal{E} \cong \frac{\alpha^2(2\mathcal{M} + \alpha)^2}{8\mathcal{M}(2n_0 + 1 + c)^2} \, , \qquad \mathcal{E}_\infty = \mathcal{M} + \alpha/2 \, . \tag{4.29}$$

This is of $O(\alpha^2)$ if $\mathcal{M} = O(1)$, and of $O(\alpha^3)$ if $\mathcal{M} = O(\alpha)$. For example, for $\mu = 1, n_0 = 0$ and $\mathcal{M} = \alpha$ one has $\Delta\mathcal{E} = 0.15\alpha^3$.

In the region $x \ll 1$, the equation becomes

$$-\frac{1}{x^2}\partial_x x^2 \partial_x \eta_{\rm s} + \left(\frac{\mu(\mu+1)}{x^2} - k_{\rm in}^2\right)\eta_{\rm s} = 0 \, , \tag{4.30}$$

with

$$k_{\rm in}^2 = -\mathcal{M}^2 + \mathcal{E}^2 + \frac{(1+\mu)}{6} \, .$$

The solutions are Bessel functions.

The numerical calculation of the eigenvalues is performed as in the previous cases by integrating the differential equation from $x = 0$ up to some $x = x_{\rm max} \gg 1$, with regular

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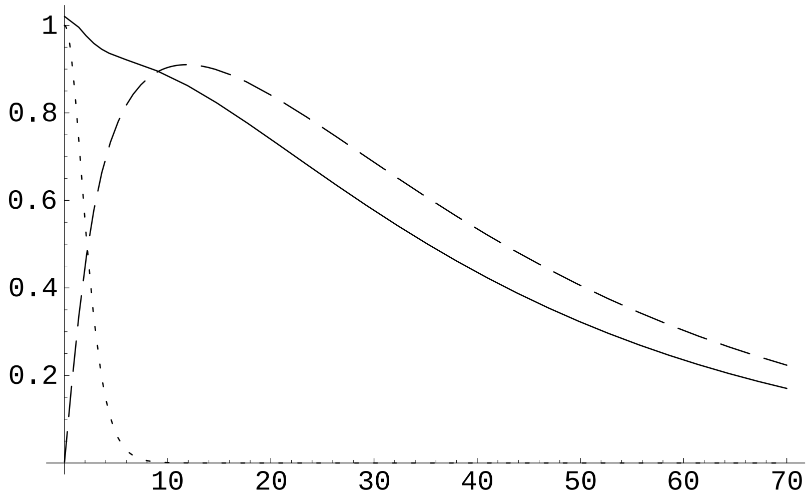

Figure 3: Wave function for the lowest energy states of $\eta_{\rm a}$ (solid line) and $\eta_{\rm s}$ (dashed line) $l = 1$, $n_0 = 0$, for $\alpha = 0.2$ and $\mathcal{M} = 0.5$.

boundary conditions at zero. The energy eigenvalues then follow from the requirement that the wavefunction approaches zero at $x = x_{\rm max}$. Some results are given in table 3.

As in the case of the previous subsection, for the states of $\eta_{\rm a}$ with $\mu = 1$ (i.e. $l = \frac{1}{2}$), the potential has no centrifugal barrier at $x = 0$, and the corresponding wave functions do not vanish at the origin. The equation in the interior $x \ll 1$ becomes

$$-\frac{1}{x^2}\partial_x x^2 \partial_x \eta_{\rm s} - k_{\rm in}^2 \eta_{\rm s} = 0\,, \tag{4.31}$$

with

$$k_{\rm in}^2 = -\mathcal{M}^2 + \mathcal{E}^2\,.$$

Although there is no centrifugal barrier near $x = 0$, it is worth noting that the particle is not concentrated at the core. Its extension grows like

$$\Delta x \cong \frac{1}{\Delta\mathcal{E}} \cong \frac{1}{\mathcal{M}\alpha^2}\,,$$

which is much greater than one for small coupling α.

The wave functions for the lowest energy states of $\eta_{\rm s}$ and $\eta_{\rm a}$ are shown in figure 3, for the case $\alpha = 0.2$, $\mathcal{M} = 0.5$. Comparing to the case of figure 2, where the Coulomb interaction is dominated by the coupling to the Higgs field, we see that the wave function is more extended. This is expected, in view of the above observation that the extension is greater for smaller couplings.

The lowest energy state for $\eta_{\rm s}$ should be unstable because the $\eta_{\rm s}$ entangled state can make a transition to a $\eta_{\rm a}$ entangled state by emission of a photon (see also the discussion in section 3). However, the lowest energy bound state for $\eta_{\rm a}$ (with $l = 1/2$, $n_0 = 0$) should be stable, since there is no possibility of decay into a state of lower energy. The same conclusion applies for the bound states of section 4.3.

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5. Summary and discussion

The main problem to which we addressed our study was the nature of the possible bound states of a scalar particle around a dyon, in systems which are globally electrically neutral. Formally, this amounts to considering the scalar field (different from the Higgs) as a fluctuation in the background of the dyon solution, and treating it in the second quantization formalism. Since we have taken as a paradigm of the dyon the Prasad-Sommerfeld solution of $N = 2$ supersymmetric SU(2) YM theory, we first considered scalar fields in $N = 2$ supermultiplets. In these cases, the scalar particle has an attractive interaction with the long range Higgs field, whose strength is fixed by the dyon solution, besides the usual Coulomb force (which is proportional to the square of the electric charge, and we naturally take it to be quite weaker than the other). In section 4.4, we considered a case where the scalar field does not interact with the Higgs, which can be the case in systems with less supersymmetry. Here the bound state is solely due to the Coulomb attraction. The main difference with respect to the previous cases is that bound states are, in this case, much more extended far from the monopole core. This is expected, since the Coulomb interaction is weaker.

It should be noted that the bound states are rather larger than the monopole core even for the cases where the Higgs attraction is dominant. Therefore they are expected to be robust against back-reaction effects of the classical dyon solution for sufficiently small coupling constant (apart from the effect of charge conservation which we have already included).

In the case of the scalar field in the adjoint representation, we have considered it to be a member of the $N = 2$ vector multiplet. Therefore it is coupled to the Higgs and it takes mass through the Higgs expectation value. The attractive force is dominated by the Higgs field and there are bound states also in the limit of vanishing charge. The bound states correspond to a certain linear combination $|\mathrm{dyon}_+\rangle|\mathrm{scalar}_-\rangle - |\mathrm{dyon}_-\rangle|\mathrm{scalar}_+\rangle$, whereas the other linear combination $|\mathrm{dyon}_+\rangle|\mathrm{scalar}_-\rangle + |\mathrm{dyon}_-\rangle|\mathrm{scalar}_+\rangle$ mixes with the state $|\mathrm{monopole}_0\rangle|\mathrm{scalar}_0\rangle$, in which both the particle and the monopole are uncharged. This mixing cannot form bound states because the neutral particle is massless and therefore it can have an arbitrarily low energy and still escape to infinity. Since some perturbation not included in our analysis, say the e.m. quanta radiation, could cause a transition from $|\mathrm{dyon}_+\rangle|\mathrm{scalar}_-\rangle - |\mathrm{dyon}_-\rangle|\mathrm{scalar}_+\rangle$ to $|\mathrm{dyon}_+\rangle|\mathrm{scalar}_-\rangle + |\mathrm{dyon}_-\rangle|\mathrm{scalar}_+\rangle$, we conclude that these bound states are unstable.

In the case of the scalar field in the fundamental representation, we have considered two cases. In the first case, the scalar is in a $N = 2$ hypermultiplet and is coupled to the Higgs field by maintaining the $N = 2$ supersymmetry of the lagrangian. We have assumed that its mass is completely due to the Higgs expectation value, in order to compare with the previous case. The resulting bound states are again essentially due to the Higgs attraction. In the second case, we have explicitly studied a SUSY breaking scenario, in which the coupling to Higgs is absent and there is an arbitrary mass parameter. The qualitative features of the two cases are rather similar.

Since in the fundamental representation there are no particles of zero charge, both combinations $|\mathrm{dyon}_+\rangle|\mathrm{scalar}_-\rangle \pm |\mathrm{dyon}_-\rangle|\mathrm{scalar}_+\rangle$ admit bound states (here we consider dyons with the same $\pm e/2$ charges as the particles in the fundamental representation, in order to have a globally neutral system). The lowest energy level is of the form $|\text{dyon}_+\rangle|\text{scalar}_-\rangle - |\text{dyon}_-\rangle|\text{scalar}_+\rangle$, with the angular momentum taking the minimum value $l = \frac{1}{2}$. In this case the "interior centrifugal barrier" is weak and the wave function is nonzero at the origin, where typically it takes its maximum value (an exception occurs in the second case with no coupling to Higgs, when the charge is very small: in such case, we found that the maximum of the wave function appears at some finite r). The extension of the wave function is in both cases larger than the monopole core size. The orbiting particle can thus be always considered to be mostly outside the core, with a wave function similar to that of the Hydrogen atom. The main role of the monopole core is to produce a full quantum entanglement, despite its small effect on the r dependence of the wave function.

Thus a bona-fide bound state picture emerges in both cases. We conclude that we get stable, quantum entangled, bound states in the case of the fundamental representation.

While generic quantum entanglement of charges may occur in various physical situations (for instance in the case of the decay of a neutral particle in two charged components), the remarkable phenomenon found here is that stable bound states of a light particle and a heavy dyon can only exist in an entangled state involving the system with opposite charge. The bound particle and the dyon can be very far from each other, and they are not charge eigenstates (although the total charge is zero). Although this setting does not seem to be in conflict with any physical law, it is nevertheless a curious effect and one could amuse her-(him-)self by imagining would be paradoxes in the chemistry of such an "atom".

Note that a measurement of the charge of the orbiting particle should make the wavefunction precipitate into a defined charge eigenstate (say, from $|\text{dyon}_+\rangle|\text{scalar}_-\rangle - |\text{dyon}_-\rangle|\text{scalar}_+\rangle$ to $|\text{dyon}_+\rangle|\text{scalar}_-\rangle$). Since such bound state is not possible, one concludes that any measurement of the charge should require an energy above the binding energy, so that the final scalar particle can escape to infinity.

Similar bound states should exist for fermions and vector particles i.e. they should give rise to a state consisting of a mixture of fermions or W^+ and W^- gauge bosons entangled with dyons. This may be more easily derived by embedding the $N = 2$ supersymmetric model in $N = 4$ super Yang-Mills theory, so that the solutions of the wave equation for fermion and gauge boson fluctuations are connected by an unbroken supersymmetry to the solutions for scalar fields computed here.

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A. Harmonic analysis for adjoint scalars

In this appendix we perform the harmonic analysis of the coupled equations for the case of scalar particles in the adjoint representation (the harmonic analysis in the case of a pointlike monopole has been done in ref. [19]).

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Using eqs. (3.6)–(3.9), we obtain the following system of equations in spherical coordinates:

$$
\begin{aligned}
\vec{D}^2 \cdot (\hat{\gamma}\, F_0(\vec{r})) &= \hat{\gamma}\left(\frac{1}{r^2}\partial_r r^2 \partial_r - \frac{2K^2}{r^2} - \frac{1}{r^2}\hat{L}^2\right) F_0 - \hat{\alpha}_- \frac{K}{r^2 \sin^2\theta}(\sin\theta\partial_\theta - i\partial_\varphi)F_0 - \\
&\quad -\hat{\alpha}_+ \frac{K}{r^2 \sin^2\theta}(\sin\theta\partial_\theta + i\partial_\varphi)F_0\,, \qquad \text{(A.1)}\\
\vec{D}^2 \cdot [\hat{\alpha}_\mp F_\pm] &= \hat{\alpha}_\mp \frac{e^{\pm i\varphi}}{\sin\theta}\left[\frac{1}{r^2}\partial_r r^2\partial_r - \frac{1}{r^2}(-1+K^2) - \frac{1}{r^2}\hat{J}^2_\pm\right] h_\pm + \hat{\gamma}\frac{2K}{r^2}\frac{e^{\pm i\varphi}}{\sin\theta}\bar{\nabla}_\pm h_\pm\,, \qquad \text{(A.2)}
\end{aligned}
$$

where

$$
\hat{L}^2 = -\frac{1}{\sin^2\theta}(\sin\theta\partial_\theta + i\partial_\varphi)(\sin\theta\partial_\theta - i\partial_\varphi)\,, \qquad \text{(A.3)}
$$

is the standard angular momentum and

$$
\hat{J}^2_+ = -\frac{1}{\sin^2\theta}\nabla_+\bar{\nabla}_+\,, \qquad \hat{J}^2_- = -\frac{1}{\sin^2\theta}\nabla_-\bar{\nabla}_-\,, \qquad \text{(A.4)}
$$

is a covariant angular momentum. Here

$$
\begin{aligned}
\nabla_\pm &= \sin\theta\partial_\theta \mp i\partial_\varphi + (1-\cos\theta)\,,\\
\bar{\nabla}_\pm &= \sin\theta\partial_\theta \pm i\partial_\varphi - (1-\cos\theta)\,,\\
[\nabla_\pm, \bar{\nabla}_\pm] &= -2\sin^2\theta\,. \qquad \text{(A.5)}
\end{aligned}
$$

The full equations of motion can now be rewritten as

$$
\begin{aligned}
&\left(-\frac{1}{r^2}\partial_r r^2\partial_r + \frac{2K^2}{r^2} + \frac{1}{r^2}\hat{L}^2\right)F_0 - \frac{2K}{r^2\sin\theta}[e^{i\varphi}\bar{\nabla}_+ h_+ + e^{-i\varphi}\bar{\nabla}_- h_-] = E^2 F_0\,, \qquad \text{(A.6)}\\
&\left[-\frac{1}{r^2}\partial_r r^2\partial_r + \frac{1}{r^2}(-1+K^2) + \frac{1}{r^2}\hat{J}^2\right] h_+ + \frac{Ke^{-i\varphi}}{r^2\sin\theta}(\sin\theta\partial_\theta - i\partial_\varphi)F_0 = \\
&\quad = [(E - eV\hat{q})^2 - e^2a^2G^2]h_+\,, \qquad \text{(A.7)}\\
&\left[-\frac{1}{r^2}\partial_r r^2\partial_r + \frac{1}{r^2}(-1+K^2) + \frac{1}{r^2}\hat{J}^2\right] h_- + \frac{Ke^{i\varphi}}{r^2\sin\theta}(\sin\theta\partial_\theta + i\partial_\varphi)F_0 = \\
&\quad = [(E + eV\hat{q})^2 - e^2a^2G^2]h_-\,. \qquad \text{(A.8)}
\end{aligned}
$$

The angular dependence is solved by setting

$$
\begin{aligned}
F_0(\vec{r}) &= \phi_0(r)Y_{lm}(\theta,\varphi)\,,\\
h_+(\vec{r}) &= \frac{1}{l_0}\phi_+(r)Z^+_{lm}(\theta,\varphi)\,, \qquad h_-(\vec{r}) = \frac{1}{l_0}\phi_-(r)Z^-_{lm}(\theta,\varphi)\,, \qquad \text{(A.9)}\\
l_0 &\equiv \sqrt{l(l+1)}\,, \qquad \text{(A.10)}
\end{aligned}
$$

where $Y_{lm}(\theta,\varphi)$ are the spherical harmonics, i.e.

$$
\hat{L}^2 Y_{lm} = l(l+1)Y_{lm}\,, \qquad \text{(A.11)}
$$

with $l = 0, 1, 2 \ldots$, and $Z^{\pm}_{lm}$ are eigenfunctions of the operator $\hat{J}^2$,

$$\hat{J}^2 Z_{lm} = l(l+1) Z_{lm} \,. \tag{A.12}$$

Explicitly, they are given by

$$Z^{\pm}_{lm} = \frac{e^{\mp i\varphi}}{\sin\theta} (\sin\theta\partial_\theta \mp i\partial_\varphi) Y_{lm} \,. \tag{A.13}$$

Using (A.13), one can check that (A.12) is indeed satisfied.

Here $l = 1, 2, \ldots$. Note that there is no eigenfunction of $\hat{J}^2$ for $l = 0$, since there are no normalizable solutions of $\bar{\nabla}_{\pm} Z_0 = 0$.

Using eqs. (A.6)–(A.9) and the properties:

$$\frac{e^{i\varphi}}{\sin\theta} \bar{\nabla}_+ Z^+_{lm} = -l(l+1) Y_{lm} \,, \qquad \frac{e^{-i\varphi}}{\sin\theta} \bar{\nabla}_- Z^-_{lm} = -l(l+1) Y_{lm} \,,$$

we find eqs. (3.12)–(3.14).

The functions $Z^{\pm}_{lm}$ (A.13) have a simple form in terms of θ, φ. For example, for $l = 1$ we have the following solutions

$$\begin{aligned}
m = 0: \qquad & F_0(\vec{r}) \;=\; \phi_0(r)\cos\theta \,, \qquad h_{\pm}(\vec{r}) = \frac{1}{\sqrt{2}} \phi_{\pm}(r) e^{\mp i\varphi} \sin\theta \\
m = 1: \qquad & F_0(\vec{r}) \;=\; \phi_0(r) e^{i\varphi} \sin\theta \,, \qquad h_+(\vec{r}) = \frac{1}{\sqrt{2}} \phi_+(r)(1 + \cos\theta) \,, \\
& h_-(\vec{r}) \;=\; \frac{1}{\sqrt{2}} \phi_-(r) e^{2i\varphi}(-1 + \cos\theta) \,, \\
m = -1: \qquad & F_0(\vec{r}) \;=\; \phi_0(r) e^{-i\varphi} \sin\theta \,, \qquad h_+(\vec{r}) = \frac{1}{\sqrt{2}} \phi_+(r) e^{-2i\varphi}(-1 + \cos\theta) \,, \\
& h_-(\vec{r}) \;=\; \frac{1}{\sqrt{2}} \phi_-(r)(1 + \cos\theta) \,.
\end{aligned} \tag{A.14}$$

Similarly, one can write down expressions for higher l.

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